Variable range hopping in the Coulomb glass

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We use a local mean-field (Hartree-like) approach to study the conductance of a strongly localized electron system in two dimensions. We find a crossover between a regime where Coulomb interactions modify the conductance significantly to a regime where they are negligible. We show that under rather general conditions the conduction obeys a scaling relation which we verify using numerical simulations. The use of a local mean-field approach gives a clear physical picture, and removes the ambiguity of the use of single-particle tunneling density of states (DOS) in the calculation of the conductance. Furthermore, the theory contains interaction-induced correlations between the on site energy of the localized states and distances, as well as finite temperature corrections of the DOS.

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I. INTRODUCTION

The unusual temperature dependence of the conductance of strongly disordered samples has generated a great interest in the past four decades. These systems manifest an interplay between disorder and interactions, which is ubiquitous in nature, and which renders the problem difficult theoretically and rich experimentally.[1–4] In these disordered systems, conductance occurs via phonon assisted hopping between localized states, whose size is roughly the localization length ξ. The measured conductance of various disordered samples shows temperature dependence different than simple activation, of the type $\sigma \sim e^{-(T_0/T)^{\zeta}}$, with $\zeta < 1$.

Mott's theory ("Mott's law") for variable range hopping (VRH),[5] which ignores the Coulomb interactions, gives $\zeta = \frac{1}{d+1}$ and T_0 equal to the noninteracting mean level spacing $\delta_\xi = 1/\nu_0 \xi^d$; with $d=1$, 2 or 3 the dimension of the system and ν_0 the density of states.

As first pointed out by Pollak,[6] interactions affect the single-particle tunneling density of states (DOS), and thus may possibly influence the conductance. Efros and Shklovskii (ES)[7] obtained an analytic bound for the so-called Coulomb gap in the DOS, which was later observed experimentally.[8,9] Using the interaction-modified DOS to find the conductance (treating the problem as essentially noninteracting from this point) one obtains ("ES's law") $\zeta = \frac{1}{2}$, in any dimension d. In ES's law T_0 is given by the Coulomb energy of the localized state $E_\xi = e^2/\xi$. Mott's law, ES's law as well as the crossover between the two regimes, were observed in various experiments.[10] Many qualitative features that this complicated system exhibits can be obtained using rather simple and heuristic arguments. However, using the single-particle DOS in the calculation of the conductance, is not discussed or cannot be justified in the framework of the heuristic arguments (see Ref. 11 for an elaborate explanation).

In this work we study the crossover between Mott's and ES's laws, using the theoretical framework of a Hartree theory.[12,13] The Hartree theory takes into account interactions in a self-consistent manner. Hence, it gives a well defined procedure for calculating various physical properties and allows us to develop a clear intuitive physical picture. Within the Hartree approximation the use of a renormalized single-particle DOS is built in, with the electrons moving in an effective potential due to the other electrons. Interaction-induced correlations between on-site energies and position[14–16] exist in the theory we present: The local mean-field equations[13] contain both the electronic site positions and the renormalized energies, and therefore, correlations between the two exist in the self-consistent solution. We emphasize that within the Hartree theory the average occupation and renormalized energy are different at each electronic site. Albeit, the Hartree method is uncontrolled as there is no small parameter in the theory. It is, therefore, difficult to estimate rigorously its limit of validity. Nevertheless, due to the long-range nature of the Coulomb interactions, the potential energy at a given site is renormalized by a large number of electronic sites. This suggests that the approximation should be valid, and we believe it should be applicable for a wide range of system parameters. Indeed, the local mean-field picture yields the well-established results for the Coulomb gap in two dimensions,[13] and captures well the results of aging experiments.[17] In this work, we demonstrate, in two dimensions, that the Hartree theory yields the ES's and Mott's laws including the crossover between them, for which we derive an analytical scaling relation that holds under generic conditions, extending the results of Refs. 18–21. This is a proof that interaction-induced correlations between location and energy as well as finite temperature corrections are not essential for determining the conductance.

The Hartree approximation involves a numerical procedure. Nevertheless, it clarifies our physical understanding of the complex Coulomb glass. Its success in reproducing equilibrium (the gap in the DOS) and near equilibrium properties (the conductance) and its simplicity indicate that it may be usable in the estimates of dynamical properties. Indeed, this was done in.[17]

The structure of the paper is as follows. We first present the model, and the heuristic arguments leading to the Mott and ES laws. We show the Hartree theory gives a Coulomb gap consistent with other theoretical and numerical approaches, and study the temperature dependence of the DOS. We then explain the local mean-field approach to determining the conductivity, and the numerical procedure involved. Finally, we analyze the crossover between the two regimes

analytically, under certain assumptions, and compare this to the numerical results. The discussion that follows is centered around the two-dimensional case, except for the scaling analysis which is done for completeness in arbitrary dimension.

II. MODEL OF HOPPING BETWEEN LOCALIZED STATES

The model analyzed consists of N localized states and $M<N$ interacting electrons, with a coupling between the electrons and a phonon reservoir.[13] The states are localized at sites r_i, $i=1..N$, and by assumption the average distance between the states is larger than the localization length ξ. The states have different on-site energies, ϵ_i, due to the disorder in the hosting lattice. For simplicity, we will neglect in our analysis fluctuations in ξ from site to site. Since the states are localized, the electrons will interact via an unscreened Coulomb potential $e^2/|r_j-r_i|$ (to take into account the dielectric constant κ, one should replace e^2 by $\frac{e^2}{\kappa}$ throughout). The disorder bandwidth W, will be assumed to be much larger than the Coulomb energy term $\frac{e^2}{r_{nn}}$, with r_{nn} the average nearest-neighbor distance. In the self-consistent Hartree approximation, the energy of the states is renormalized from ϵ_i to an energy E_i due to the interaction with the other electrons.[13]

Let us denote the energy difference of the electronic system before and after a phonon-induced tunneling event by $E_{ij}=E_j-E_i$, and the distance between their locations by $r_{ij}=|r_j-r_i|$. For weak electron-phonon coupling the transition rate γ_{ij}^0 of an electron from site i to site j, is approximately (for $E_{ij}<0$):[1]

$$\gamma_{ij}^0 \sim |M_q|^2 \nu_p f(E_i)[1-f(E_j)]e^{-r_{ij}/\xi}[1+n(|E_{ij}|)], \qquad (1)$$

where $f(E)$ and $n(E)$ are the Fermi-Dirac and the Bose-Einstein distributions, respectively; M_q the electron-phonon coupling constant and ν_p the phononic density of states. For upward transitions ($E_{ij}>0$) the square brackets are replaced by $n(|E_{ij}|)$. We emphasize that the energy difference contains the Coulomb interaction with all other charges. The e^2/r_{ij} interaction term present in the Efros-Shklovskii argument should not be included here, as in the local mean-field (rate equations) approach the interactions are already taken into account (albeit approximately but self-consistently and with good results). It may also be said that charge is continuously transferred. Indeed, only without this term will the approach lead to detailed balance at equilibrium and to the correct result for the Coulomb gap in two dimensions,[13] including the coefficient $\frac{2}{\pi e^4}$, as demonstrated in Fig. 1. Recent Monte Carlo simulations have confirmed the expected parabolic form of the three dimensional Coulomb gap,[22] and it would be interesting to see this also for the two dimensional case.

III. ANALYSIS USING HEURISTIC ARGUMENTS

The heuristic arguments are based upon the low temperature approximation that the conductance follows:[1]

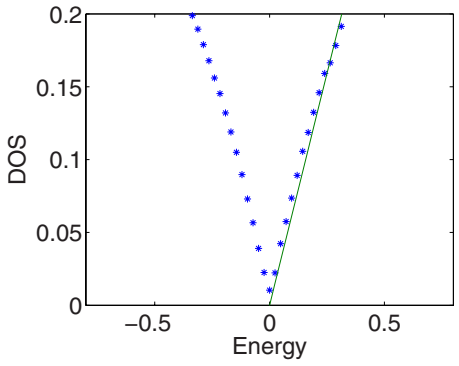

FIG. 1. (Color online) A comparison between the Coulomb gap obtained by the local mean-field approximation, in two dimensions, and the theoretical prediction[23] of a linear gap with slope $\frac{2}{\pi e^4}$. The number of sites used was N=10 000, and the Fermi energy was $E_f=0$ (half filling). The sites were uniformly distributed in a square, with energy scale $e^2 n^{1/2}=1$, where n is the density. The temperature was taken to be zero. The on-site disorder was distributed uniformly in the interval $[-\frac{1}{2},\frac{1}{2}]$. The y axis denotes the probability density of the energies E_i. The graph is the average over 300 instances.

$$\sigma \propto \gamma_{ij}^0 \propto e^{-(r_{ij}/\xi-|E_{ij}|/T)}. \qquad (2)$$

Where γ_{ij}^0 are two optimally coupled sites (i.e, with the highest rate between them). Indeed, it pays sometimes for the electron to hop a larger distance, thereby finding a state closer in energy and reducing the necessary inelastic energy transfer.[5]

In the local mean-field treatment it is legitimate to use the single-particle concept of DOS and to ask how interactions modify it. To determine the shape of the interacting DOS, $\nu(E)$, (not to be confused with the noninteracting DOS ν_0) we require that the noninteracting and interacting energy-distance relation are equal, i.e., for two dimensions $E=e^2/r=1/\nu(E)r^2$. This yields the ES result $\nu(e)\sim E/e^4$. The tunneling DOS is suppressed at low energies due to interactions.[6,7,24–26] Notice that $\nu(E)\sim\nu_0$ at $E_C=e^4\nu_0$, an energy above which the DOS becomes ν_0. Hence, E_C determines the scale of the Coulomb gap.

Regarding the conductivity of the system, *a priori* there could be four energy scales involved: the bandwidth W; the Coulomb energy $\frac{e^2}{r_{nn}}$, with r_{nn} the average nearest-neighbor distance; the Coulomb energy over the localization length distance $E_\xi\equiv\frac{e^2}{\xi}$; and $\delta_\xi\equiv\frac{1}{\nu_0\xi^2}$, the level spacing in a box of size ξ. However, it is clear that the conductance as well as other physical quantities cannot depend on W. Even in the limit of $W\to\infty$, while keeping the (noninteracting) DOS at the Fermi level, ν_0, constant, the expressions should be finite, since the energies involved in transport processes should occur near the Fermi energy E_F. Since $\frac{e^2}{r_{nn}}=\frac{E_\xi\sqrt{W}}{\sqrt{\delta_\xi}}$, this scale is also ruled out. Indeed, it is possible to express all the physical properties using E_ξ and δ_ξ. Using similar arguments we can understand the formula for E_C: Since the DOS is a static property of the system, determined by interaction energies on scales much larger than the localization length ξ, it cannot depend on ξ. The only energy scale that we can construct from E_ξ and δ_ξ that does not depend on ξ is $E_\xi^2/\delta_\xi=E_C$.

Optimizing Eq. (2) with the DOS described above yields Mott's law for temperatures above a certain temperature and ES's law[1] for lower temperatures. The crossover temperature in two dimensions is given by:

$$T_x = E_\xi^3 / \delta_\xi^2. \tag{3}$$

Note that this is consistent with the physical picture given previously, where it was shown that all physical properties must depend on E_ξ and δ_ξ.

So far we have reviewed simple heuristic arguments and discussed Mott's and ES's laws in a non-rigorous way. This enabled us to introduce the energy and length scale in the problem we study. In the rest of the paper, we will perform analytical and numerical analysis of the local mean-field approach, and show that it reproduces the nonrigorous results although it includes temperature and correlation effects that were omitted in the above heuristic approach.

IV. TEMPERATURE DEPENDENCE OF THE COULOMB GAP

Determining the value of the finite DOS at the Fermi energy due to the effect of temperature is a problem which has not been settled yet: numerical investigations by[24] seem to be in disagreement with other theoretical predictions.[27] We hereby present the results of the local mean-field theory. We calculate the DOS at the Fermi energy in the presence Coulomb interactions and finite temperature. For an infinite system at zero temperature, the Efros-Shklovskii Coulomb gap will make the DOS vanish at the Fermi energy. We know that finite-size effects will give rise to a finite DOS, proportional to $\frac{1}{\sqrt{N}}$.[12] Let us assume we have a system large enough so that these finite-size effects will be negligible compared to the effects of the finite temperature. The finite-temperature local mean-field equations are:[13]

$$E_i = \epsilon_i + \sum_{j \neq i} \left(\frac{1}{1 + e^{(E_j - E_F)/T}} - Q_b \right) \frac{e^2}{r_{ij}}. \tag{4}$$

For half filling, $E_F = 0$, and the positive background charge $Q_b = 0.5$.

We have solved these equations numerically in two dimensions, and found a linear dependence of the DOS at E_F as function of temperature. This is consistent with Refs. 27 and 28. The results are demonstrated in Fig. 2. Thus, $\nu(E_f) = \alpha T$, and since the Coulomb gap is washed out at a temperature of order E_C, we know that $\alpha \sim \frac{\nu_0}{E_C}$. For the data corresponding to Fig. 2 we find that $\alpha \approx 0.15$. In this case the ratio $\frac{\nu_0}{E_C} \approx 0.1$, which is consistent. We can now understand why this temperature dependence does not affect the crossover discussed in the previous section: while the finite-temperature contribution is linear in T, the optimal hop energy scales as $\sqrt{T}$ at low temperatures (before the crossover to Mott's regime). One can check that even at the crossover temperature the finite-temperature correction to the DOS is not important.

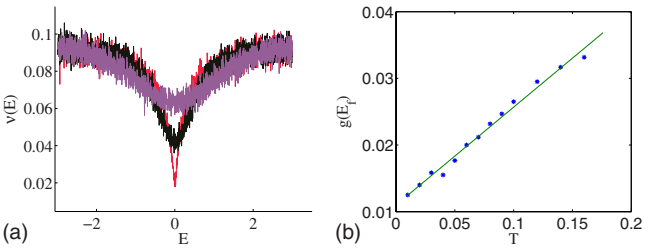

FIG. 2. (Color online) Density of states as a function of energy and temperature. Each graph is an average over 2000 iterations. For each, $N = 1000$, and the ratio $\frac{T}{e^2 n^{1/2}}$, where T is the temperature, and n the density, was varied between 0.01 and 0.58. The on-site disorder was taken uniform in the interval $[-5, 5]$.

V. LOCAL MEAN-FIELD CALCULATION OF THE CONDUCTANCE

We calculated the conductance numerically using the following scheme:

(1) Find the equilibrium occupation number and energy associated with each electronic site within the local mean-field picture, using Eq. (4). This takes the interactions into account, and the density of states obtained will manifest the Coulomb gap.[13]

(2) Construct the Miller-Abrahams resistor network.[29,30] For a noninteracting problem, this is an *exact* method to find the resistance of the system by finding that of a certain resistor network. Here, the resistor between the sites i and j is given by $R_{ij} = \frac{kT}{e^2 \gamma_{ij}^0}$, where γ_{ij}^0 are the equilibrium transition rates, which are calculated based on the energies and occupation numbers of the sites, see Eq. (1). Two sites which are close to each other in space and close in energy to the Fermi energy, will have a lower value of resistance between them: if the sites are far away in space, the overlap integrals will vanish exponentially. If the sites are far from the Fermi energy, they will be permanently full or empty, and will not contribute to the conductance. The Appendix derives the resistor network equations starting from the local mean-field approach.

(3) Find the resistance of the system. This is done by solving for the energies (voltages) at the sites. In steady state, the sum of currents into and out of each site must vanish. Expressing these currents in terms of the resistor network values and the energies yields a set of linear equations, solved numerically.

Executing the numerical procedure, we found that above a certain crossover temperature T_x, given by Eq. (3), approximately $e^{-(T_0/T)^\zeta}$ behavior with $\zeta \sim 0.34 \pm 0.01$ was observed for the resistivity, close to the Mott value in two dimensions of $\frac{1}{3}$. Below this temperature $\zeta \sim 0.49 \pm 0.02$ was observed, in accordance with the Efros-Shklvoskii result.[31] In the next section, we give a scaling argument, which shows that:

$$\log \sigma \sim \delta_\xi / E_\xi f(T_x / T). \tag{5}$$

This allows us to collapse numerical data obtained for different sets of parameters. The crossover between the Mott and Efros-Shklovskii regimes is manifested in the asymptotics of $f(x)$. Figure 3 demonstrates the crossover between the two

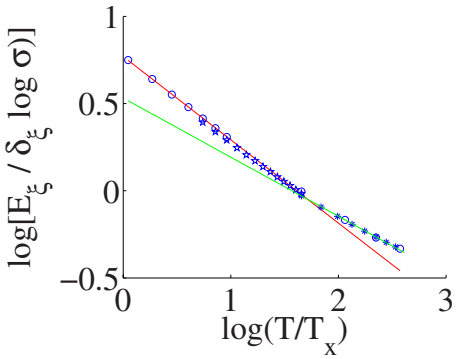

FIG. 3. (Color online) The crossover between ES and Mott VRH as the temperature is varied. Using the scaling relation of Eq. (5), we have collapsed data from different parameter sets onto the same master curve. We use N between 500 and 3000, and each point represents the average over 10 runs. We have verified that logarithmic averaging yields essentially the same results. The sites were chosen with uniform probability in a $\sqrt{N}X\sqrt{N}$ square with density $n=1$, and the energy scale was chosen such that $e^2n^{1/2}=1$. $n^{1/2}\xi$ was varied between 0.1 and 0.2, where ξ is the localization length, and the on-site disorder was distributed uniformly in the interval $[-\frac{1}{2},\frac{1}{2}]$. The temperature axis was scaled according to the crossover temperature of Eq. (3). The conductance was scaled according to the scaling relation of Eq. (14) The lower temperature fit has an exponent of 0.49 ± 0.02, consistent with the Efros-Shklovskii value of 1/2 while the higher temperature fit has an exponent of 0.34 ± 0.01, consistent with the Mott value of 1/3. The lines are the best fits for the relevant data sets. In the Mott regime where the linear fit was taken conductance changes by more than 3 orders of magnitude. In the ES regime, conductance changes by more than 6 orders of magnitude.

regimes, and the data collapse, validating the scaling relation. We should emphasize, however, that the scaling relation is derived under certain assumptions: we assume that the temperature is low enough such that we do not have the trivial scenario of nearest-neighbor hopping. At $T>W$, we shall have nearest-neighbor hopping, and weak dependence of the conductance on temperature: this gives a "smoothing off" of the Mott exponent, to a temperature independent regime. Thus, we do not expect complete data collapse for the regime where the temperature is larger than the disorder. Additionally, at very low temperatures, finite-size effects give rise to a finite DOS, and thus give rise to deviations from the Efros-Shklovskii hopping mechanism. Nevertheless, Fig. 3 shows clearly that data collapse is obtained. The least-squares fits in the asymptotic (linear) regimes gave the exponents discussed previously.

VI. FORM OF THE CROSSOVER

The crossover between the Mott and ES regimes was discussed in Refs. 18–21. We shall now calculate the form of the crossover within our theoretical framework. We shall show that scaling is obeyed, for any dimensions, regardless of the details of the Coulomb gap. This generalizes previous work, for any DOS which depends only on the parameters e^2 and ν_0. As previously explained, this is the generic form for the DOS. Notice that in principle the DOS is weakly dependent on temperature, and therefore this scaling relation is only an approximation. Let us proceed to the proof, for any dimension d. The basic equation connecting the energy E and distance r associated with a hop is obtained by requiring that the average number of sites within an area of order r^2 and energy lower than E should be of order unity. Denoting the DOS as $\nu(E)$, this gives:

$$r^2\int_0^E \nu(E')dE' \sim 1. \tag{6}$$

The second equation comes from optimizing $\eta=\frac{r}{\xi}+\frac{E}{T}$. The optimal η will determine the conductance of the system via $\log\sigma\sim-\eta$. Differentiating the first equation with respect to r gives

$$\nu(E)=\frac{-d}{r^{d+1}}\frac{dr}{dE}. \tag{7}$$

Demanding the derivative of η with respect to r to vanish gives

$$\frac{1}{\xi}+\frac{1}{T}\frac{dE}{dr}=0. \tag{8}$$

Combining the last two equations yields

$$\nu(E)=\frac{\xi d}{[r(E)]^{d+1}T}, \tag{9}$$

therefore:

$$\int^E \nu(E')dE'=\left[\frac{\nu(E)T}{d\xi}\right]^{d/(d+1)}. \tag{10}$$

To proceed, it is useful to define the function $\psi(E)=\frac{\int^E\nu(E')dE'}{[\frac{\nu(E)}{2}]^{d/(d+1)}}$. We have by assumption $\psi=\psi(E,\nu_0,e^2)$, and we obtain the equation:

$$\psi(E,\nu_0,e^2)=\left(\frac{T}{\xi}\right)^{d/(d+1)}. \tag{11}$$

Since the dimensions of the LHS are $[\text{energy/length}]^{d/(d+1)}$, ψ must have the form: $\psi=g(m)[E^{2/d}\nu_0^{2/d}e^2]^{d/(d+1)}$, where $m=\frac{E}{e^{2d/(d-1)}\nu_0^{1/(d-1)}}$ is a dimensionless combination of the parameters of ψ, and $g(m)$ is a certain function. Then the equation takes the form:

$$\left(\frac{E}{e^{2d/(d-1)}\nu_0^{1/(d-1)}}\right)=g^{-1}\left[\left(\frac{T}{E^{2/d}\nu_0^{2/d}e^2\xi}\right)^{d/(d+1)}\right]. \tag{12}$$

We have an equation of the form $AE=f(BE)$, with $A=\frac{1}{e^{2d/(d-1)}\nu_0^{1/(d-1)}}$ and $B=e^d\nu_0(\frac{\xi}{T})^{d/2}$. Both AE and BE are dimensionless. The general solution will give us $E=\frac{1}{A}\phi(\frac{B}{A})$, for a certain function ϕ. Plugging in the values of A and B, we obtain

$$\frac{E}{T}=\frac{1}{(e^2\nu_0)^{1/(d-1)}\xi}\theta\left(\frac{T_x}{T}\right), \tag{13}$$

with T_x as defined by Eq. (3), and θ is a nonuniversal function, depending on the form of the DOS.

A similar dimensional analysis shows that $\frac{r}{\xi}$ follows an identical scaling law (but with a different function). Using the definition of η, we reach the conclusion that the conductance follows the proposed scaling, which can be written in terms of the energy scales δ_ξ and E_ξ as:

$$\log \sigma \sim \left(\frac{\delta_\xi}{E_\xi}\right) f[(E_\xi^{d+1}/\delta_\xi^2)^{d-1}/T]. \tag{14}$$

Notice that for $d=2$ we retrieve the crossover temperature of Eq. (3). The analysis did not assume anything on the shape of the DOS, other than the assumption that it is a function of ν_0 and e^2: it does not have to contain a plateau or a power-law Coulomb gap. Nevertheless, if it does contain these features, f must have asymptotics which correspond to the Mott and ES VRH: at $x \gg 1$ we have $f \propto x^{1/(d+1)}$, while for $x \ll 1$ we have $f \propto x^{1/2}$. An advantage of this calculation compared to other theories is that it allows one to complete the calculation for *any* DOS. Hence, it can be used by experimentalists to "reverse-engineer" and find the function f that described the DOS from the conductivity measurements.

VII. SUMMARY

We have given a consistent Hartree-theory for the conductance of the electron glass. The long-range nature of the interactions should justify this approximation. In a previous work we showed the theory gives a linear Coulomb gap at zero temperature,[13] in accordance with other theories. Since the interactions are treated on a local mean-field level, we can still use the single-particle DOS to characterize the conductance. This explains why the Coulomb gap indeed affects the conductance. Indeed, by numerically solving the Miller-Abrahams resistor network for the system, we found a crossover between the ES and Mott VRH regimes. Scaling is generically obeyed at the crossover, which we have derived analytically (neglecting interaction-induced correlations), and compared to the numerics (which takes them into account). In future work we intend to incorporate many-electron tunneling into the theory, as well as study the out-of-equilibrium properties of the system.

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APPENDIX: MILLER-ABRAHAMS RESISTOR NETWORK WITHIN THE LOCAL MEAN-FIELD APPROACH

In this appendix we show how one can find the current distribution in a near-equilibrium steady state, and hence the conductance, by solving a linear resistor problem. The resistance between sites i and j is inversely proportional to the equilibrium current between them. This is a generalization of the concept of the Miller-Abrahams resistor network,[29] since here the on-site energies are modified due to the Coulomb interactions, taken on a Hartree level. The whole derivation follows, without any assumptions, from the local mean-field equations.[13] A similar treatment of the subject can be found in.[1] Here, we also discuss the effective resistor between a site and the leads, which is not usually discussed.

The local mean-field equations for a steady-state in the presence of an external field are obtained by equating the time-derivative to zero:

$$\sum_{j\neq i} \gamma_{ij} = \sum_{j\neq i} \gamma_{ji}, \tag{A1}$$

with $\gamma_{ij}=M_{ij}n_i(1-n_j)[N(|\Delta E|)+\theta(\Delta E)]$. Here, $M_{i,j}$ is related to an overlap integral and is therefore symmetric. ΔE is the energy difference before and after a tunneling event. In fact, one should also add the contributions of the currents to the leads, but we postpone the discussion of this to the end of the appendix. Within the framework of the local mean-field approximation, each electron moves within a renormalized effective potential, and therefore the on-site energies are given by:

$$E_i = \epsilon_i + \sum_{j\neq i} (n_i - Q_b)e^2/r_{ij}, \tag{A2}$$

where ϵ_i is the on-site energy, containing the "bare" disorder and the external electric potential, and Q_b is the background charge.

The energy difference $\delta_E=E_i-E_j$, does not contain the usual $-e^2/r_{ij}$ term, as in the seminal derivation of the Coulomb gap by Efros and Shklovskii.[7] This is an important point distinguishing the local mean-field approaches from other methods: within the local mean-field approximation, the charge at each site is a statistical ensemble-average, and is therefore a continuous number between 0 and 1. The equations of the time-evolution of the site occupations *on the local mean-field level* do not contain explicitly the $-e^2/r_{ij}$ term in the energy difference: the interactions were already taken into account, in a self-consistent way. Intuitively, one may think of a continuous transfer of charge, where obviously this term should not be present.

It should be noted that there are approaches that go beyond the mean-field picture, and treat the correlations effect (in an approximate way) (see for example, Ref. 1 p. 239). An advantage of the local mean-field approach, is that it yields Fermi-Dirac statistics for the site energies: it is easy to see that requiring Fermi-Dirac statistics gives detailed balance, i.e., $n_i(1-n_j)[N(\Delta E)+1]=n_j(1-n_i)[N(\Delta E)]$, if $\Delta E=E_i-E_j$, but this does not follow if we includes the $-e^2/r_{ij}$ term. It was shown in this manuscript that Eq. (A2) together with the requirement of Fermi-Dirac statistics yields a linear Coulomb gap in two-dimensions, with a slope of $2/\pi$ consistent with other works, which is an important validity test for the local mean-field approximation.

1. Resistors between two sites in the bulk

Continuing from Eq. (A1), assuming the electric field small we can linearize the expression for γ_{ij} near the equilibrium values, and utilize the fact that detailed balance is

obtained at equilibrium, making the constant term vanish.

Writing the occupations in the steady-state as $n_i = n_i^0 + \delta n_i$, this results in the following equation for the current between sites i and j, assuming $E_i > E_j$:

$$\delta\gamma_{ij} = \frac{\gamma_{ij}^0}{n_i^0}\delta n_i - \frac{\gamma_{ij}^0}{1-n_j^0}\delta n_j + \frac{\gamma_{ij}^0}{N+1}\frac{\partial N}{\partial E}\delta\Delta E. \quad \text{(A3)}$$

Thus,

$$\delta\gamma_{ij} = \frac{\gamma_{ij}^0}{n_i^0}\delta n_i - \frac{\gamma_{ij}^0}{1-n_j^0}\delta n_j - N\gamma_{ij}^0(\delta E_i - \delta E_j)/T. \quad \text{(A4)}$$

Similarly we obtain

$$\delta\gamma_{ji} = \frac{\gamma_{ij}^0}{n_j^0}\delta n_j - \frac{\gamma_{ij}^0}{1-n_i^0}\delta n_i - (N+1)\gamma_{ij}^0(\delta E_i - \delta E_j)/T. \quad \text{(A5)}$$

It is useful to define local electrochemical potentials by demanding Fermi-Dirac statistics to be obeyed locally with respect to them, i.e., $n_i = 1/(1+\exp[(E_i-\mu_i)/T])$. Thus:

$$\delta n_i = n_i^0(1-n_i^0)(\delta\mu_i - \delta E_i)/T. \quad \text{(A6)}$$

Combining Eqs. (A1) and (A4)–(A6) we obtain:

$$\sum_{j\neq i}\frac{\gamma_{ij}^0}{T}[(\mu_i - \delta E_i) - (\mu_j - \delta E_j) + (\delta E_i - \delta E_j)] = 0. \quad \text{(A7)}$$

So finally we have:

$$\sum_{j\neq i}\frac{\gamma_{ij}^0}{T}[\mu_i - \mu_j] = 0. \quad \text{(A8)}$$

2. Boundary conditions

The boundary conditions are given by the voltage drop across the sample, dictating μ_L and μ_R, the right and left chemical potential. Equation (A8) together with this boundary conditions are exactly equivalent to Kirchhoff's law, if we define a conductance between i and j as γ_{ij}^0/T. This is precisely the Miller-Abrahams resistor network, albeit with interactions which renormalize the values of each resistor (since the on-site energies are renormalized by the interactions, so are the rates).

3. Resistors between a site in the bulk and the leads

For completeness, we discuss the role played by the coupling between the sites and the leads, which is often overlooked at. Indeed, if one is interested in the bulk properties (or if one measures the conductance using a four-terminal measurement) then this 'contact resistance' will not play an important role. However, for small samples this may be an important effect, and one should also take care of the finite-size effects when performing numerical simulations. Unlike the case of tunneling from site to site that have discrete energies, the leads have a continuum spectrum. Therefore tunneling to the leads is possible also without absorbing or emitting a phonon, and we neglect the modification of the tunneling to the leads due to the phonons. Using Fermi's golden rule, we have for the tunneling rate, say from the left lead to a site i:

$$\Gamma_{L,i} \sim n^{FD}(E_i - [\mu + \delta\mu])(1-n_i), \quad \text{(A9)}$$

where n^{FD} is the Fermi-Dirac function (since the lead is assumed to be in thermal equilibrium at chemical potential $\mu + \delta\mu$). In equilibrium, $\delta\mu = 0$, and $n^{FD}(E_i^0 - \mu) = n_i^0$. Upon linearizing near the equilibrium configuration we obtain:

$$\delta\Gamma_{L,i} \sim -(1-n_i^0)\Gamma_{L,i}^0(\delta E_i - \delta\mu)/T - \Gamma_{L,i}^0/(1-n_i^0)\delta n_i, \quad \text{(A10)}$$

where $\Gamma_{L,i}^0 \sim n_i^0(1-n_i^0)$ is the equilibrium current. Using Eq. (A6) we obtain:

$$\delta\Gamma_{L,i} \sim -(1-n_i^0)\Gamma_{L,i}^0(\delta E_i - \delta\mu)/T - n_i^0\Gamma_{L,i}^0(\delta\mu_i - \delta E_i)/T. \quad \text{(A11)}$$

In a similar fashion we may calculate $\delta\Gamma_{i,L}$, and obtain:

$$\delta\Gamma_{i,L} \sim n_i^0\Gamma_{L,i}^0(\delta E_i - \delta\mu)/T + (1-n_i^0)\Gamma_{L,i}^0(\delta\mu_i - \delta E_i)/T. \quad \text{(A12)}$$

Finally, we have that the total current between site i and the lead is given by:

$$I_{i,L} = (\delta\mu_i - \delta\mu)/R_i, \quad \text{(A13)}$$

where $R_i^{-1} = \Gamma_{L,i}^0/T$ is the effective resistance between the site and the lead. This completes the derivation of the resistor network.

[1] B. Shklovskii and A. Efros, *Electronic Properties of Doped Semiconductors* (Springer-Verlag, Berlin, 1984).

[2] *Electron-electron Interactions in Disordered Systems*, edited by A. L. Efros and M. Pollak (North-Holland, Amsterdam, 1985).

[3] V. F. Gantmakher, *Electrons and Disorder in Solids* (Oxford University Press, Oxford, 2005).

[4] *Hopping Transport in Solids*, edited by M. Pollak and B. I. Shklovskii (Elsevier, London, 1990).

[5] N. F. Mott, Philos. Mag. **19**, 835 (1969).

[6] M. Pollak, Discuss. Faraday Soc. **50**, 13 (1970).

[7] A. L. Efros and B. I. Shklovskii, J. Phys. C **8**, L49 (1975).

[8] J. G. Massey and M. Lee, Phys. Rev. Lett. **75**, 4266 (1995).

[9] V. Y. Butko, J. F. DiTusa, and P. W. Adams, Phys. Rev. Lett. **84**, 1543 (2000).

[10] A. Mobius, D. Elefant, A. Heinrich, R. Muller, J. Schumann, H. Vinzelberg, and G. Zies, J. Phys. C **16**, 6491 (1983); A. Mobius, H. Vinzelberg, C. Gladun, A. Heinrich, D. Elefant, J. Schumann, and G. Zies, *ibid.* **18**, 3337 (1985); A. Mobius, *ibid.* **18**, 4639 (1985); W. N. Shafarman, D. W. Koon, and T. G. Castner, Phys. Rev. B **40**, 1216 (1989); Y. Zhang, P. Dai, M. Levy, and M. P.

Sarachik, Phys. Rev. Lett. **64**, 2687 (1990); R. Rosenbaum, Phys. Rev. B. **44**, 3599 (1991); Y. Liu, B. Nease, K. A. McGreer, and A. M. Goldman, Europhys. Lett. **19**, 409 (1992); I. Shlimak, M. Kaveh, M. Yosefin, M. Lea, and P. Fozooni, Phys. Rev. Lett. **68**, 3076 (1992); U. Kabasawa, Y. Tarutani, M. Okamoto, T. Fukazawa, A. Tsukamoto, M. Hiratani, and K. Takagi, *ibid.* **70**, 1700 (1993); J. Lam, M. D'Iorio, D. Brown, and H. Lafontaine, Phys. Rev. B **56**, R12741 (1997); M. P. Sarachik and P. Dai, Europhys. Lett. **59**, 100 (2002).

[11] M. Pollak, Phys. Status Solidi C **5**, 667 (2008).

[12] M. Grunewald, B. Pohlmann, L. Schweitzer, and D. Wurtz, J. Phys. C **15**, L1153 (1982).

[13] A. Amir, Y. Oreg, and Y. Imry, Phys. Rev. B **77**, 165207 (2008).

[14] A. Efros, N. V. Lien, and B. Shklovskii, Solid State Commun. **32**, 851 (1979).

[15] Y. M. Blanter and M. E. Raikh, Phys. Rev. B **63**, 075304 (2001).

[16] D. N. Tsigankov and A. L. Efros, Phys. Rev. Lett. **88**, 176602 (2002).

[17] A. Amir, Y. Oreg, and Y. Imry, Phys. Rev. Lett. **103**, 126403 (2009).

[18] O. Entin-Wohlman, Y. Gefen, and Y. Shapira, J. Phys. C **16**, 1161 (1983).

[19] Y. Meir, Phys. Rev. Lett. **77**, 5265 (1996).

[20] A. Aharony, Y. Zhang, and M. P. Sarachik, Phys. Rev. Lett. **68**, 3900 (1992).

[21] R. Rosenbaum, N. V. Lien, M. R. Graham, and M. Witcomb, J. Phys.: Condens. Matter **9**, 6247 (1997).

[22] M. Goethe and M. Palassini, Phys. Rev. Lett. **103**, 045702 (2009).

[23] A. L. Efros, J. Phys. C **9**, 2021 (1976).

[24] M. Sarvestani, M. Schreiber, and T. Vojta, Phys. Rev. B **52**, R3820 (1995).

[25] F. Epperlein, M. Schreiber, and T. Vojta, Phys. Rev. B **56**, 5890 (1997).

[26] M. Müller and S. Pankov, Phys. Rev. B **75**, 144201 (2007).

[27] A. A. Mogilyanskii and M. E. Raikh, Zh. Eksp. Teor. Fiz. **95**, 1870 (1989) [Sov. Phys. JETP **68**, 1081 (1989)].

[28] E. I. Levin, V. L. Nguen, B. I. Shklovskii, and A. L. Efros, Zh. Eksp. Teor. Fiz. **92**, 1499 (1987) [Sov. Phys. JETP **65**, 842 (1987)].

[29] A. Miller and E. Abrahams, Phys. Rev. **120**, 745 (1960).

[30] V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).

[31] At low temperatures finite size effects will give rise to a finite DOS at the Fermi energy, modifying this picture and raising the conductance. These were also seen numerically.